Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00075-1

J. Appl. Maths Mechs, Vol. 63, No. 4, pp. 603–616, 1999

0021-8928/99/$—see front matter

THE PLANE CONTACT PROBLEM OF SEEPAGE CONSOLIDATION†

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(*Received* 26 *August* 1998)

Seepage consolidation due to a rigid cylinder rolling over the surface of an elastic saturated porous half-space is considered. The dependence of the moment of rolling friction on the roller velocity is found. The standard consolidation model is modified to allow for the two-phase nature of the discharge zone.

Although there is an extensive literature on contact problems of the theory of elasticity and viscoelasticity (see [1, 2], for example), very little research has been done on problems of saturated porous media which have a similar formulation in the scheme of seepage consolidation. These are, however, interesting in both mathematical and mechanical terms. On the one hand, the analogy with an abstract viscoelastic material is not so close that known methods can be used directly: special methods need to be devised. On the other, the nature of the subject gives rise to effects which do not appear at all in viscoelasticity theory. These include, for instance, the appearance of two-phase zones in the initial totally saturated porous material [3].

The same applies in full measure to the problem of the stress-strain state of a half-space under the action of a rigid cylinder rolling freely over its surface. The conventional problem here is to compute the moment of rolling friction. We know [4] that, apart from Reynolds' universal mechanism of friction [5], connected with the relative slip of the touching surfaces due to their deformation, the moment of friction also owes its appearance to the viscous properties of the base material. In the case of a porous medium, the latter are determined by the seepage return flow of fluid and, as a rule, the corresponding viscous mechanism of friction is more important than the Reynolds mechanism (see Section 6).

Sections 1–6 deal in detail with the viscous mechanism of friction in the case where the standard linear model of seepage consolidation is used to describe strains of the porous base. Although it gives a fairly accurate description of the force parameters of the process, the model has the drawback that it does not take into account the two-phase nature (liquid plus gas) of the discharge zone behind the roller. The corresponding refined model and the accompanying calculations are given in Section 7.

1. STATEMENT OF THE PROBLEM

Let an infinitely long roller of radius R move from right to left with velocity V over the surface of a saturated porous half-space.

Seepage consolidation of the half-space can be described in a moving system of coordinates associated with the roller by the equations [6, 7]

$$\mu\Delta u_x + (\lambda+\mu)\frac{\partial\theta}{\partial x} - \frac{\partial p}{\partial x} = 0 \tag{1.1}$$

$$\mu\Delta u_y + (\lambda+\mu)\frac{\partial\theta}{\partial y} - \frac{\partial p}{\partial y} = 0 \tag{1.2}$$

$$V\frac{\partial\theta}{\partial x} - k\Delta p = 0 \tag{1.3}$$

Here u_x, u_y are the displacements, p is the pressure of the fluid, λ and μ are the Lamé coefficients of the elastic porous matrix, $\theta = \operatorname{div} u$ is the volume deformation of the medium and k is the seepage coefficient, determined by the permeability k_0 of the porous medium and the viscosity μ_0 of the fluid

†*Prikl. Mat. Mekh.* Vol. 63, No. 4, pp. 629–644, 1999.

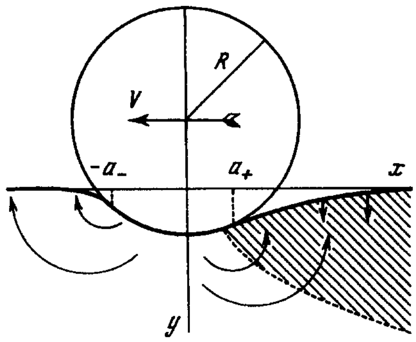

Fig. 1.

which saturates it ($k = k_0/\mu_0$). The compressibility of the grains of the skeleton and fluid has been neglected in Eq. (1.3). In this case the volume macro-deformations (the first term in (1.3)) are associated with repacking of the grains and are uniquely defined by the change of pore volume due to the fluid being squeezed out of there (the second term in (1.3)). The effective stresses [6] are linearly related to the strains

$$\sigma_{xx} = \lambda\theta + 2\mu\frac{\partial u_x}{\partial x}, \quad \sigma_{yy} = \lambda\theta + 2\mu\frac{\partial u_y}{\partial y}, \quad \sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{1.4}$$

With the usual assumption of small strains, the boundary conditions are specified on the line $y = 0$. They are such that

$$y = 0; \; x > a_+, \; x < -a_-: \quad \sigma_{yy} = \sigma_{xy} = p = 0 \tag{1.5}$$

outside the contact area, and

$$y = 0; \; -a_- < x < a_+: \; \frac{\partial p}{\partial y} = 0, \; \frac{\partial u_y}{\partial x} = -\frac{x}{R}, \; \sigma_{xy} = 0 \tag{1.6}$$

inside it.

The first condition of (1.6) implies that the roller is impermeable to fluid and the second corresponds to equal normal displacements in the contact zone. The other boundary relation will describe the conditions of friction in the contact area. The last equation in (1.6) corresponds to the limiting case of a smooth roller with zero coefficient of friction between it and the material of the porous matrix [2, p. 280]. On the one hand, the use of this condition enables us to examine the mechanism of viscous friction in "pure" form: the Reynolds mechanism does not operate for a smooth roller, for which rolling friction is associated only with the viscous properties of the base. On the other hand, in contact rolling problems as a rule, the tangential forces are considerably less than the normal forces (by an order of magnitude [2]), which means that this condition is a poor approximation even in the general case.

It will be more convenient if we move the origin of coordinates to the centre of the segment ($-a_-$, a_+) and normalize the spatial coordinates on $a = (a_+ + a_-)/2$ while keeping the same notation as before. The contact area in the $z = x + iy$ plane then corresponds to the segment (–1, 1). We will also normalize the displacements to the quantity a^2/R, and the stresses and pressure to $2\mu a/R$, again without any change of notation.

We will introduce three new functions to replace the displacements and pressure. The function

$$f(z) = \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) + i\left(\frac{\theta}{r} - 2p\right) \tag{1.7}$$

is analytic, while the other two, θ and $\zeta = 2u_x - y\,\mathrm{Re}f$, satisfy the equations

$$2s\frac{\partial\theta}{\partial x} = \Delta\theta, \quad \Delta\zeta = 2\frac{\partial\theta}{\partial x} \tag{1.8}$$

The dimensionless parameters r and s are defined by the relations

$$r=\frac{\mu}{\lambda+2\mu},\quad s=\frac{aV}{2k(\lambda+2\mu)}$$

In terms of the functions f, ζ, θ the boundary conditions (1.5) and (1.6) become

$$y=0,\ |x|>1:\ \frac{\partial\zeta}{\partial y}=0,\ \theta=r\frac{\partial\zeta}{\partial x},\quad r\operatorname{Im}f=\theta \tag{1.9}$$

$$y=0,\ |x|<1:\ \frac{\partial\zeta}{\partial y}=0,\ \frac{\partial\theta}{\partial y}=2r,\quad r\operatorname{Re}f=2(x-A);\ A=\frac{a_{-}-a_{+}}{2a} \tag{1.10}$$

The constant A in (1.10), arising from the asymmetric position of the contact area relative to the centre of the roller, will be defined below.

In addition to θ and ζ, we shall consider the linear combination $\zeta-\theta/s$ which, according to (1.8), is a harmonic function. We will denote the analytic function whose imaginary part it is by $h(z)$.

Now consider the extra conditions imposed on the required functions. Obviously θ, f and dh/dz must vanish at infinity. It will become clear that h satisfies an even stronger regularity condition: both the function h itself and dh/dz tend to zero as $z\to\infty$. The conditions of continuity of strains require continuity of $z=\pm1$ at the singular points θ, f and $\nabla\zeta$. It should be emphasized that the derivatives of h and θ can (and in fact do) separately have integrable singularities at $z=\pm1$; only their linear combination ξ has to be continuously differentiable.

We have thus reduced the initial problem (1.1), (1.6) to problem (1.7), (1.8) of simultaneously finding functions θ, ζ, f and the value A from boundary conditions (1.9) and (1.10). The size of the contact area a plays an implicit role in the formulation of the problem. We normally know Q, the pressure on the base per unit length of the roller, rather than a, which is computed after solving problem (1.7)–(1.10) by integrating the normal stress p_s over the contact area. The latter is the difference between the effective normal stress $p_c=\sigma_{yy}$ transmitted over the porous matrix and the pressure $p_w=p$ of the fluid

$$p_s=p_c-p_w,\quad p_w(x)=\frac{1}{2}\left(\frac{\theta}{r}-\operatorname{Im}f\right)\bigg|_{y=0},\quad p_c(x)=\frac{1}{2}\left(\frac{\theta}{r}-\frac{\partial\zeta}{\partial x}\right)\bigg|_{y=0}$$

The corresponding integrals

$$q=q_w-q_c=-\int_{-1}^{1}p_s(x)dx,\quad m=\int_{-1}^{1}(x-A)p_s(x)dx \tag{1.11}$$

determine [1, 4] the force Q and moment of rolling friction M per unit length of the roller

$$Q=\frac{2\mu a^2}{R}q(s,r),\quad M=\frac{2\mu a^3}{R}m(s,r) \tag{1.12}$$

2. SOLUTION

The problem is split into two. We first find θ and ζ independently of f and A by solving Eqs (1.8) with the first two boundary conditions in (1.9) and (1.10). We then determine the right-hand side in the last of conditions (1.9) and thereby complete the statement of the problem of finding f and A.

We will give a brief description of the solution of these problems, referring to the qualitative properties of the required functions when necessary. The proof will be given in Section 3.

It is better to reduce the problem of finding θ and ζ to an integral equation over the part of the boundary $y=0$, $|x|>1$ relative to $g(x)=(\partial\theta/dy)(x,0)$. To do so, using the first relations in (1.9) and (1.10), we express θ and $\operatorname{Im}h$ in terms of g by the formulae

$$\begin{aligned}\operatorname{Im}h&=-\frac{1}{\pi s}\int_{-\infty}^{+\infty}g(x')\ln|z-x'|dx'\\ \theta&=-\frac{1}{\pi}\int_{-\infty}^{+\infty}g(x')e^{s(x-x')}K_0(s|z-x'|)dx'\end{aligned} \tag{2.1}$$

where K_0 is the MacDonald function. Then, taking into account the continuous differentiability of ζ and using the second pair of boundary conditions (1.9) and (1.10), we obtain the required integral equation in the form

$$\int\limits_{|x'|>1} K(s(x-x'))g(x')dx' = -2r\int\limits_{|x'|<1} K(s(x-x))dx', \quad |x|>1 \tag{2.2}$$

$$K(x;r) = e^x K_0(|x|) - r\frac{d}{dx}(\ln|x| + e^x K_0(|x|))$$

Its kernel $K(x)$ has a logarithmic singularity at zero and decays at infinity. Equation (2.2) is solved by the collocation method with multiplicative separation of the singularities g. The latter have the form

$$\begin{gathered} g = O(|x \mp 1|^{-1/2}), \quad x \to \pm 1 \\ g = O(x^{-3/2}) \quad x \to +\infty; \quad g = O(x^{-2}) \quad x \to -\infty \end{gathered} \tag{2.3}$$

By separating the singularities we can obtain acceptable results even on fairly coarse grids.

Having found g, the values of Imh and θ on the boundary $y = 0$ which are needed later are calculated by simply integrating (2.1). Incidentally, there is no need to do this for $x > 1$. It can be shown (see Section 3) that

$$\theta(x,0) = \zeta(x,0) = \operatorname{Im} h(x,0) = 0, \quad x>1 \tag{2.4}$$

This is equivalent to the fact that, as well as σ_{yy} and σ_{xy}, σ_{xx} vanishes on the free surface behind the roller. This can be understood in physical terms: there are no mechanisms for the contraction or extension of the surface layer behind the roller in the x direction.

We now consider the problem of finding f and A when the function $\theta(x, 0)$ is known. This is Signorini's problem with a known solution [8]. The solvability condition gives

$$A = \frac{1}{2r\pi}\int\limits_{-\infty}^{-1} \frac{\theta(x,0)dx}{\sqrt{x^2-1}} \tag{2.5}$$

and the required function is found from the formula

$$f = 2z - 2\sqrt{z^2-1} - 2A - \frac{\sqrt{z^2-1}}{\pi r}\int\limits_{-\infty}^{-1} \frac{\theta(x',0)}{\sqrt{x'^2-1}}\frac{dx'}{x'-z} \tag{2.6}$$

taking the positive branch of $\sqrt{(z^2-1)}$ on the segment $(1, \infty)$. Note that since the function $\theta(x, 0)$ is bounded at $x = -1$ and decreases rapidly at infinity

$$\theta(x,0) = O(|x|^{-3/2}), \quad x \to -\infty \tag{2.7}$$

the integrals in (2.5) and (2.6) are defined.

3. QUALITATIVE ANALYSIS IF THE PROBLEM. THE BEHAVIOUR OF THE SOLUTION AT INFINITY

In the qualitative analysis of problem (1.8)–(1.10), we consider the following auxiliary problem: Eqs (1.8) at $s = 1$ with boundary conditions

$$y=0: \quad \frac{\partial\zeta}{\partial y} = 0, \quad \frac{\theta}{r} - \frac{\partial\zeta}{\partial x} = -\delta(x) \quad (\operatorname{Im} h = \zeta - \theta) \tag{3.1}$$

where the Dirac function δ describes the normal point load on the porous base matrix. This solution is denoted by a zero subscript. The solution of the initial problem θ, h is represented in terms of θ_0, h_0 and the effective normal stresses p_c on the contact area as follows:

$$\theta = -2s\int\limits_{-1}^{1} p_c(x')\theta_0(s(z-x'))dx', \quad h = -2\int\limits_{-1}^{1} p_c(x')h_0(s(z-x'))dx' \tag{3.2}$$

The auxiliary problem can be solved formally by using a Fourier transformation with respect to x. In particular, for values θ_0 on the boundary we obtain

$$\theta_0(x,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\upsilon x}\left(b(i\upsilon)-\frac{r}{1-r}\right)d\upsilon \tag{3.3}$$

$$b(w)=\frac{r}{1-r}-r\left(1+wr\left(\sqrt{1-\frac{2}{w}}-1\right)\right)^{-1} \tag{3.4}$$

The square root in (3.4) is defined in the complex plane $w = u + i\upsilon$ with a non-negative real part on the imaginary axis. This condition will be satisfied if w is cut on the segment (0, 2) and the branch of the root which is greater than zero on the interval $(2, \infty)$ is chosen. Then (3.3) will take the form

$$\theta_0(x,0)=-\frac{r}{1-r}\delta(x)+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{wx}b(w)dw$$

The function $b(w)$ has no singularities in the left half-plane and tends to zero like w^{-1} as $w \to \infty$. By Jordan's lemma [9] it follows that $\theta_0(x, 0) = 0$ for $x > 0$. Relation (2.4) is an obvious consequence of that fact.

Similarly, for negative x, we can replace integration over the imaginary axis by integration over the cut (0, 2). As a result we obtain

$$\theta_0(x,0)=E'(x),\quad E(x)=\frac{r^2}{\pi}\int_0^2\frac{e^{ux}\sqrt{2-u}}{1-2r(1-r)u}\frac{du}{u^{1/2}},\quad x<0 \tag{3.5}$$

There are some properties of the function $E(x)$ that should be noted. Together with all its derivatives, it increases monotonely. The way in which they behave at infinity depends on the contribution to the integral (3.5) from the left-hand end, so that

$$E\sim E_{-1}\,|\,x\,|^{-1/2},\quad E_{-1}=r^2\sqrt{2/\pi},\quad x\to\infty \tag{3.6}$$

and this relation can be differentiated any number of times. The values at zero are found by direct integration of (3.5)

$$E(0)=E_0=\frac{r^2}{1-r},\quad E'(0)=E_1=\frac{r^2}{2(1-r)^2},\ \ldots$$

Thus on the boundary $y = 0$ we have

$$\theta_0(x,0)=-\frac{r}{1-r}\delta(x)+\begin{cases}0, & x>0\\ E', & x<0\end{cases}$$

From this and (3.1) we find in succession the boundary values of ζ_0 and imaginary part h_0. The function $h_0(z)$ is recovered in the entire half-plane from the values of the axes of the imaginary part on the boundary by the Schwarz integral

$$h_0=-\frac{r}{\pi(1-r)z}+\frac{1}{\pi}\int_{-\infty}^{0}\left(\frac{1}{r}E-E'\right)(x')\frac{dx'}{x'-z}$$

The function is expanded in series in powers of $z^{-1/2}$ at an infinitely distant point, the principal term of the series being determined by the nature of the decay of $E(x)$ at infinity

$$h_0(z)=-E_{-1}r^{-1}z^{-1/2}+O(z^{-1}),\quad z\to\infty \tag{3.7}$$

A similar relation is obtained for θ_0 from the representation

$$\theta_0(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\theta_0(x',0)\frac{ye^{(x-x')}}{|\,z-x'\,|}K_1(|\,z-x'\,|)dx'$$

an analogue of the Schwarz integral.

The principal term of the respective series is determined by the value $E_0 - r/(1-r) = -r$ of the integral over the boundary of $\theta_0(x, 0)$. Using the well-known asymptotic form at infinity of the modified Bessel function K_1, we find

$$\theta_0(z) \sim -\frac{r}{\sqrt{2\pi}} e^{-(|z|-x)} \frac{y}{|z|^{3/2}}, \quad |z| \to \infty \tag{3.8}$$

By virtue of (3.2), the behaviour (3.7) and (3.8) of the functions θ_0 and h_0 at infinity applies to θ and h also. All that is needed is to replace θ_0, h_0 and z in (3.7) and (3.8) by $\theta/(2sq_c)$, $h/(2sq_c)$ and zs, respectively.

As for the function f, it follows from (2.5) and (2.6) that it can be represented as $z \to \infty$, apart from terms of order z^{-3}, in the form

$$f(z) = \frac{1}{z} - \frac{1}{\pi r} \int_{-\infty}^{-1} \frac{\theta(x',0)}{\sqrt{x'^2 - 1}} \frac{x'dx'}{x' - z}$$

and thus decreases as z^{-1} for large z. Since θ and h vanish at infinity, the coefficient of the respective terms is numerically equal to the load transmitted to the base by the roller, that is

$$f \sim \frac{2q}{\pi z}, \quad q = \frac{\pi}{2} + \frac{1}{r} \int_{-\infty}^{-1} \frac{\theta(x,0)}{\sqrt{x^2 - 1}} x dx \tag{3.9}$$

The way in which the stresses σ, the pressure p and the associated seepage flow ∇p change at infinity can be found from (3.7)–(3.9).

The entire half-space can be split into two regions. The first is the boundary layer of discharge next to the boundary behind the contact area (the dashed region in Fig. 1). Its depth varies as the square root of the distance from the contact area.

The second region is the rest of the half-plane, in which h and θ are much smaller than f, and everything depends on the behaviour (3.9) of the latter. The tensor $\sigma^0 = \sigma - p\delta^{ij}$ of total stresses is the same as in the Boussinesq solution [1] of point loading on an elastic half-space. Its principal axes are the axes of the polar system of coordinates $\rho = |z|$, $\phi = \arg z$, where

$$\sigma^0_{\rho\rho} = -\frac{2q_s \sin\phi}{\pi\rho}, \quad \sigma^0_{\rho\phi} = \sigma^0_{\phi\phi} = 0$$

For the effective stresses we have

$$\sigma_{\rho\rho} = -\sigma_{\phi\phi} = -p = -\frac{q_s \sin\phi}{\pi\rho}$$

so that, in some sense, the load at infinity is distributed equally between the fluid and the porous matrix. The direction of the seepage flow, determined by the pressure gradient, is from the contact area to the free surface. The fluid moves over a family of semi-circles and is extruded onto the surface in front of the roller ($x < 0$), reaching the discharge region behind it ($x > 0$).

Another part of the fluid arrives from the surface into the discharge region, inside which, in boundary-layer coordinates ρ, $\tilde{\phi} = \phi\sqrt{(\rho s)}$, we have

$$p \approx \frac{\theta}{2r} \approx -\frac{q_c}{\sqrt{2\pi\rho}} \tilde{\phi} \exp\left(-\frac{\tilde{\phi}^2}{2}\right)$$

It is clear from physical considerations that $q_c > 0$ and, therefore, the pressure in the discharge region is negative; it decreases as $\tilde{\phi}$ increases from zero on the surface to its minimum value $\sim -\rho^{-1}$ at the centre of the discharge region at $\tilde{\phi} = 1$ and rapidly tends to zero as $\tilde{\phi}$ increases further. The fluid moves away from the surface and from within the half-space towards the centre of the discharge region, the flow from the surface, of order $\rho^{-3/2}$, being asymptotically greater than the flow $q_s/\pi\rho^2$ from within. The arrows in Fig. 1 indicate the overall pattern of motion of the fluid.

As before, the principal axes for the stresses in the discharge region are ρ and φ (or, equivalently here, x and y) and $\sigma_{\rho\rho} = (1-2r)p$, $\sigma_{\phi\phi} = p$.

4. A RAPIDLY MOVING ROLLER

The principal term of the asymptotic representation of the solution as $s \to \infty$

$$\theta \sim 0, \quad h \sim 0, \quad A \sim 0, \quad f \sim 2(z - \sqrt{z^2 - 1}) \tag{4.1}$$

does not involve the volume deformation θ of the porous medium. This is understandable. If the roller is moving rapidly, the fluid does not have time to filter through and the porous base effectively becomes incompressible.

Relations (4.1) and (1.11) indicate that the contact area and the stresses on it are symmetric relative to the centre of the roller, the moment of rolling friction is zero, and the entire load at the contact between the roller and the porous base is absorbed by the fluid

$$q \sim q_w \sim \pi/2, \quad q_c \sim m \sim 0$$

To find out how $m(s)$ and $A(s)$ decrease as $s \to \infty$ we must find the next terms of the asymptotic series. Apart from higher-order small terms, they are

$$\theta = s^{-1/2}\theta^0(x, ys^{1/2}), \quad h = s^{-1}h^0, \quad f = 2(z - \sqrt{z^2 - 1}) + s^{-1}f^0$$

$$p_c = s^{-1/2}p_c^0(x), \quad A = s^{-1}A^0, \quad m = s^{-1}m^0 \tag{4.2}$$

We should point out that the change of θ is of boundary-layer character. The contribution of θ to the stresses and strains is small ($s^{-1/2}$ compared with unity). The corresponding term is significant only for seepage flow near the contact area and in the discharge region, where its contribution is of the same order ($O(1)$) as in the principal term found from (4.1), "correcting" the discrepancy between the latter and boundary condition (1.6) with respect to the seepage flow. Without dwelling on the determination of θ^0, we shall confine ourselves to finding f^0 and h^0. This will be enough for us to be able to compute A^0 and m^0.

From (3.2) and (3.7) we have

$$h^0(x) = 2\sqrt{\frac{2}{\pi}}\int_{-1}^{1}\frac{p_c^0(x')dx'}{\sqrt{z - x'}}$$

It follows that the real and imaginary parts of h^0 are equal to zero on the segments $(-\infty, -1)$ and $(1, \infty)$, respectively. On $(-1, 1)$, according to (1.10), the real part dh^0/dz is constant and equal to $-r$. Thus, we find dh^0/dz by solving a mixed boundary-value problem for a function which is analytic in the upper half-plane and vanishes at infinity under the boundary conditions

$$y = 0 \quad \operatorname{Re}\frac{dh^0}{dz} = -2r, \quad |x| < 1; \quad \operatorname{Re}\frac{dh^0}{dz} = 0, \quad x < -1; \quad \operatorname{Im}\frac{dh^0}{dz} = 0, \quad x > 1$$

Since h^0 is bounded, this problem is uniquely solvable and has a solution of the form

$$\frac{dh^0}{dz} = \frac{r}{\pi i}\left(\ln\frac{\sqrt{z-1} - \sqrt{2i}}{\sqrt{z-1} + \sqrt{2i}} + \frac{2\sqrt{2i}}{\sqrt{z-1}}\right)$$

The real branches of the root and logarithm on the positive semi-axis are taken here.

It follows from boundary conditions (1.9) and (1.10) that the imaginary part f^0 coincides with the imaginary part $(1, \infty)$ on $(-\infty, -1)$ and dh^0/dz, and the real part f^0 is equal to $(-1, 1)$ on $-2A^0$. The difference $f^1 = f^0 - dh^0/dz$ is therefore given by

$$y = 0: \quad \operatorname{Im} f^1 = 0, \quad |x| > 1; \quad \operatorname{Re} f^1 = 2(r - A^0), \quad |x| < 1$$

Since f^0 is continuous at $z = 1$, the function f^1 will have the same singularity at that point as $-dh^0/dz$. Moreover, it will be continuous at $z = -1$ and vanish at infinity. A^0 and f^1 are uniquely defined by these conditions

$$A^0 = r\frac{\pi - 2}{\pi}, \quad f^1 = \frac{2r}{\pi}\left(1 - \sqrt{\frac{z+1}{z-1}}\right)$$

after which m^0 is computed by integrating $\operatorname{Im} f^1$ over the contact area $(-1, 1)$.

As a result we obtain $m^0 = \pi r/2$ and, finally,

$$q \approx \frac{\pi}{2}, \quad A \approx r\frac{\pi - 2}{\pi s}, \quad m \approx \frac{\pi r}{2s}, \quad s \to \infty \tag{4.3}$$

In dimensional variables

$$a \approx \sqrt{\frac{QR}{\pi\mu}}, \quad M \approx \frac{2k\mu Q}{V}, \quad V \to \infty$$

As expected, the shear modulus μ, and not λ, appears in the final formulae.

The asymptotic behaviour of the solution is in fact more complex than shown by formulae (4.2). Near the ends of the segment $(-1, 1)$ the function θ has internal exponential boundary layers. Taking these into account has no influence on the required integral characteristics of the solution, affecting the behaviour of $h^0 = f^0$ in only a small neighbourhood of the point $z = -1$. The weak logarithmic singularity (instead of continuity) of f^0 at $z = -1$ is a consequence of this fact.

5. A SLOWLY MOVING ROLLER

For a roller at rest ($s = 0$) the pressure p is identically equal to zero, and the initial problem (1.1)–(1.6) becomes the classical contact problem [1] of the theory of elasticity. Its solution in terms of functions θ, ζ, f has the form

$$\operatorname{Im} f = R, \quad \theta = rR, \quad \frac{\partial\zeta}{dx} = r\left(1 - y\frac{\partial}{\partial y}\right)R, \quad \frac{\partial\zeta}{\partial y} = ry\frac{\partial R}{\partial x}, \quad R = 2\operatorname{Im}(z - \sqrt{z^2 - 1}) \tag{5.1}$$

As in the case of a fast-moving roller, the contact area and stresses on it are symmetric about the centre of the roller and the friction moment is equal to zero. However, the load at the contact is borne by the porous matrix rather than the fluid

$$p_s = p_c = \frac{1-r}{2}R(x, 0), \quad q = q_c = \frac{1-r}{2}\pi, \quad q_w = m = 0 \tag{5.2}$$

The behaviour of the friction moment $m(s)$ and of the asymmetry $A(s)$ of the contact area for small s is of interest. We will compute the value of A, by finding the function $\theta(z)$ in the interval $(-\infty, -1)$. For small s, by virtue of (3.2) and the first relation in (5.2), in the principal term we have

$$\theta(x, 0) = -s(1-r)\int_{-1}^{1} R(x', 0)E'(s(x - x'))dx', \quad x < -1 \tag{5.3}$$

Using representation (3.5) for E, we reduce (5.3) to the form

$$\begin{aligned}\theta(x, 0) &= \frac{2sr^2(1-r)}{\pi}\int_0^2 e^{sux} f(u)\int_{-1}^{1} \sqrt{1 - x'^2}e^{-sux'}dx'du \approx \\ &\approx sr^2(1-r)\int_0^2 e^{sux} f(u)du; \quad f(u) = \frac{\sqrt{u(2-u)}}{1 - 2r(1-r)u}\end{aligned} \tag{5.4}$$

It is valid to replace the exponential in the inner integral by one for small s since ux' is bounded.

It remains to use relation (2.5) for A. Making the substitution (5.4) and changing the order of integration, we obtain

$$A \approx \frac{sr(1-r)}{2\pi}\int_0^2 f(u)\int_{-\infty}^{-1} e^{sux}\frac{dx}{\sqrt{x^2 - 1}}du$$

The inner integral is calculated explicitly [10, p. 323] and is equal to $K_0(su)$. Therefore, for small s, it is equal to $-(\gamma + \ln(us/2))$, where γ is Euler's constant. The final expression for A is thus

$$A \approx \frac{sr}{4(1-r)}\left(\ln\frac{1}{s}+A_1\right),\quad s\to 0; \tag{5.5}$$

$$A_1 = \frac{2}{\pi}(1-r)^2\int_0^2 f(u)\left(\ln\frac{2}{u}-\gamma\right)du$$

As r varies from zero to 1/2, $A_1(r)$ decreases from 2ln2 to 2ln2 − γ − 1. The intermediate values of A_1 can be given here

r	0	0.1	0.2	0.25	0.3	0.35	0.4	0.45	0.5
$A_1\times 10^3$	309	238	156	111	61	7	−52	−118	−191

The friction moment m behaves similarly. Omitting the calculations, we give the final result

$$m \approx \frac{\pi rs}{8}\left(\ln\frac{1}{s}+A_2\right),\quad A_2 = A_1+\frac{5}{4},\quad s\to 0 \tag{5.6}$$

6. ANALYSIS OF THE RESULTS

We have seen from the preceding sections that in analysing the relations between q and m and the dimensionless velocity of the roller s, it is better to normalize the total load and friction moment as follows

$$\tilde{q} = \frac{1}{r}\left(1-\frac{2}{\pi}q\right),\quad \tilde{m} = \frac{3m}{r}$$

The way in which $\tilde{q}(s)$ and $\tilde{m}(s)$ behave at zero and at infinity is independent of r. Moreover, it turns out that $\tilde{q}$, $\tilde{m}$ for different r are close to one another over the entire range of variation of s. This is illustrated in Fig. 2, which shows the limiting dependences of $\tilde{q}$ and $\tilde{m}$ on s at $r = 0$ (the solid curves) and $r = 0.5$ (the dashed curves). The curves for $0 < r < 0.5$ lie in between these.

The variable q appears in formula (1.12), which relates the length a of the contact area to the applied load Q. The fact that q is a function of s shows that a depends on the velocity of the roller V as well as the load Q. However, the relation between a and V is weak and can often be neglected. In fact, q increases only slightly, from $(1 - r)\pi/2$ to $\pi/2$, as V increases from zero to infinity. Taking its average value, we obtain

$$a \approx \sqrt{\frac{QR}{\pi\mu(1-r/2)}} \tag{6.1}$$

With a defined in this way, the relative error is never more than 22%, and typically $r \sim$ 0.3–10%.

The dependence $\tilde{m}(s)$ is, essentially, the dependence of the friction moment on the velocity with the

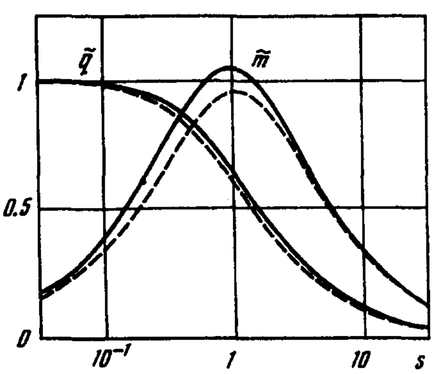

Fig. 2.

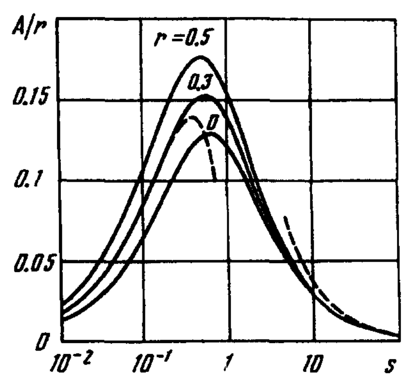

Fig. 3.

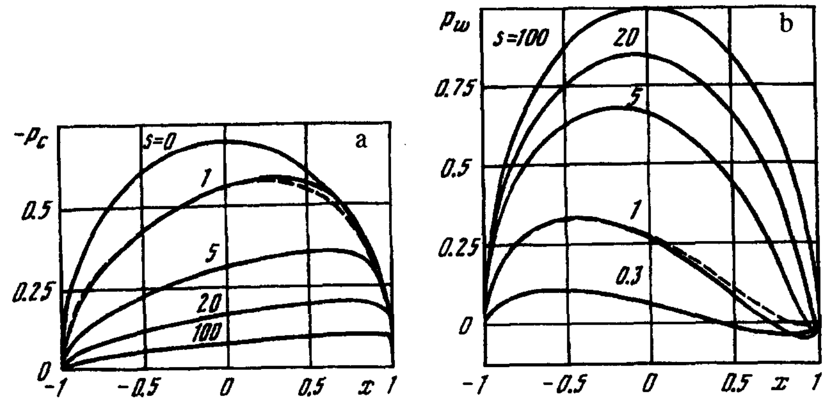

Fig. 4.

other parameters fixed. Figure 2 shows that the relation is non-monotone. The rolling friction is a maximum at the characteristic velocity

$$V_* \approx 2k(\lambda + 2\mu)/a \tag{6.2}$$

The friction disappears as the velocity increases to infinity or decreases to zero. This is a general feature of this mechanism [1, 2, 4].

The asymmetry of the contact area $A(s)$ behaves in the same way as the velocity changes. This can be seen from Fig. 3, which shows A/r as a function of s for different r. The asymptotic forms obtained for $r = 0.3$ in Section 5 and 6 are represented by the dashed lines. Even the maximum possible value of $A(s; r)$ at $r = 0.5$, $s = 0.56$, is very small, at only 0.088.

We will compare the values for the viscous and Reynolds mechanisms, neglecting their mutual influence for simplicity. Consider the dimensionless coefficient of rolling friction $F = M/(arQ)$. For fixed r, F depends only on the dimensionless velocity of the roller s for the viscous mechanism, but it depends on the coefficient of sliding friction [2] for the Reynolds mechanism. In both cases F reaches its maximum at intermediate values of the respective parameter. This maximum (Fig. 2) is about 0.2 for the viscous mechanism, and [2, p. 286] 1.5×10^{-3} for the Reynolds mechanism, a difference of two orders of magnitude. In the general case of saturated porous media, therefore, the viscous mechanism predominates over the Reynolds mechanism.

The asymptotic analysis of the solution in Sections 4 and 5 has shown that as the dimensionless velocity s increases from zero to infinity, the effective stresses $p_c(x)$ on the contact area decrease in absolute magnitude from $(1 - r)\sqrt{(1 - x^2)}$ to zero, and the pressure $p_w(x)$ of the fluid increases from zero to $\sqrt{(1 - x^2)}$. The solid lines in Fig. 4 give some idea of how these functions change for intermediate values of s. The calculations were carried out for $r = 0.3$ for different values of s.

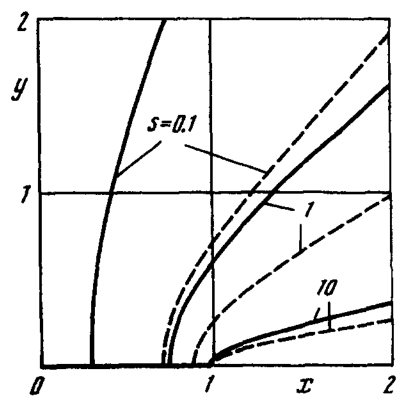

Fig. 5.

It is interesting to note the segment of negative pressure at the back of the contact area. This is consistent with what happened in the discharge region in Section 3. The region of negative pressures, which starts at the contact area, extends along the free surface to infinity. The boundary of the region is indicated by the solid lines in Fig. 5. Here too the calculations were carried out for $r = 0.3$ for different values of s. As s increases the boundary approaches the free surface. Simultaneously the region of negative pressure on the contact area contracts towards its end.

The presence of a region of negative pressures is an interesting feature of the problem (though not at all rare in the theory of seepage consolidation [3]). We recall that the pressure is measured relative to atmospheric pressure. In seepage theory it is usual to attribute any fall in pressure additional to atmospheric pressure in regions near the free surface to the fact that their pore space is occupied by a two-phase medium (water plus air) [11, 12]. This is not taken into account in the classical model of seepage consolidation used in this study. An adequate description of the processes in the discharge region would require an appropriate refinement of that model.

7. REFINEMENT OF THE MODEL

One generalization of the model which allows for the above fact is given in [3]. Using that generalization, we represent the equation of mass balance of the fluid (1.3) in a form which is valid in the region of both total and partial saturation

$$V\left(S\frac{\partial\theta}{\partial x}+m\frac{\partial S}{\partial x}\right)-k\nabla\ \varkappa\nabla p=0 \tag{7.1}$$

Here S is the water-saturation, m is the porosity and $\varkappa(p)$ is the phase permeability with respect to the fluid of the under-saturated porous medium. In the region of total saturation $S = 1$, $\varkappa = 1$ and (7.1) becomes the same as (1.3). In the case of a coarse-grained medium or for large external loads, we can use the limit dependences of S and $\varkappa$ on p

$$\varkappa = S \in H(p) \tag{7.2}$$

where H is the Heaviside function, and the inclusion denotes that S belongs to the graph of the corresponding multivalued function. These assumptions enable us to neglect the influence of the capillary pressure on the effective stresses and thereby use Eqs (1.1) and (1.2) without change.

Thus, (1.1)–(1.3) is replaced by (1.1), (1.2), (7.1) and (7.2). The differential inclusion (7.1), (7.2) distinguishes two subregions of space. In one (the seepage zone) $p > 0$, $S = 1$ and the initial model (1.1)–(1.3) remains true. In the other (the two-phase zone) $0 < S < 1, p = 0$, and (1.3) is replaced by the relation

$$\partial/\partial x(\theta+m\ln S)=0 \tag{7.3}$$

which defines the saturation there.

On the boundary γ between these zones the pressure and normal flow of the fluid must be continuous

$$\gamma:\ \ p=0,\ \ Vm(S_- -1)n_x+k\partial p_+/\partial n=0 \tag{7.4}$$

Here n is the normal to γ towards the seepage zone, the plus and minus subscripts denoting the seepage an two-phase zones respectively.

Condition (7.4) can be simplified further in the case being considered here, because the boundary $\gamma : x = x_0(y)$ corresponds to the monotonely increasing function x_0, so that in (7.4) $n_x < 0$. But since p is positive in the seepage zone, from the first condition of (7.4) the value of $\partial p_+/\partial x$ in (7.4) is non-negative. This is possible only when simultaneously $\partial p_+/\partial x = 0$ and $S_- = 1$ on γ. The first of these conditions means that the problem of finding p and γ is independent of that of finding S as a classical variational inequality (see [13], for example)

$$V\frac{\partial\theta}{\partial x}=\Delta p,\ \ p\geqslant 0 \tag{7.5}$$

Once the corresponding problem has been solved, the second condition together with (7.3) gives the saturation in the two-phase zone

$$x>x_0(y):\ \ m\ln S=\theta(x,\ y)-\theta(x_0,\ y) \tag{7.6}$$

Note that since θ is small, the value of S is close to unity and $\ln S$ in (7.6) can be replaced by $(S - 1)$.

We will briefly describe the main points of the numerical solution of problem (1.1), (1.2), (7.5). Grid methods are used. First, the region $\operatorname{Im} z > 0$ is mapped into a unit semi-circle, in which we change to polar coordinates $0 < \rho < 1.0 < \varphi < \pi$

$$w = \rho e^{i\varphi}, \quad z = -(w + w^{-1})/2$$

The original problem in terms of the functions $f_i = \operatorname{Im} f$, θ, ζ then takes the form

$$-\Delta f_i = 0 \tag{7.7}$$

$$-\Delta\theta + sL\theta = 0, \quad \theta \geqslant rf_i, \tag{7.8}$$

$$-\Delta\zeta + L\theta = 0 \tag{7.9}$$

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}, \quad L = \frac{1-\rho^2}{\rho^2}\cos\varphi\frac{\partial}{\partial\rho} + \frac{1+\rho^2}{\rho^3}\sin\varphi\frac{\partial}{\partial\varphi}$$

The boundary conditions can be written as follows

$$\rho = 1: \frac{\partial f_i}{\partial\rho} = -2\sin\varphi, \quad \frac{\partial\theta}{\partial\rho} = r\frac{\partial f_i}{\partial\rho}, \quad \frac{\partial\zeta}{\partial\rho} = 0 \tag{7.10}$$

$$\varphi = \frac{\pi}{2} \mp \frac{\pi}{2}: f_i = \pm\frac{2r\rho^2}{1-\rho^2}\frac{\partial\zeta}{\partial\rho}, \quad \theta = rf_i, \quad \frac{\partial\zeta}{\partial\varphi} = 0 \tag{7.11}$$

Note that A does not appear here. It only arises in connection with $\operatorname{Re} f$ and can be found after solving problem (7.7)–(7.11), using relation (2.5) for instance.

Equations (7.7)–(7.9) with boundary conditions (7.10) and (7.11) are solved by iteration. The computations at each step are as follows. Starting with the value of ζ found in the previous iteration on the boundary, f_i is found from Eq. (7.7) for the first pair of boundary conditions in (7.10) and (7.11). Then θ is found from (7.8) for the second boundary conditions in (7.10) and (7.11). Finally, the value of ζ is refined by solving Eq. (7.9) for the last pair of boundary conditions (7.10) and (7.11).

The first boundary condition in (7.11) and convective terms in (7.9) are approximated using central differences, the convective terms in (7.8) using differences against the flow, and the Laplace operator in the standard way throughout [14]. We use a uniform grid for φ in every case, and for ρ for moderate and large s. If s is small, there is a boundary layer at the origin of coordinates, so that the grid for ρ must be compressed accordingly and the coordinates of its nodes vary quadratically.

The fact that the grid is uniform with respect to φ means that we can find f_i and ζ using the fast methods of expansion in a single series employing a discrete Fourier transformation [14]. The values of θ are found by a method of point upper relaxation and their simultaneous projection onto an admissible set. It is unnecessary to find θ exactly at each step. A better strategy is to perform a fixed number of iterations of the upper relaxation method on each step.

Typically the calculations were carried out on a 64×64 grid, with 100 upper relaxation iterations at each step. In no case did the total number of steps exceed 20.

The results were tested on a classical model—in this case it is obviously necessary only to exclude the projection when finding θ. The accuracy of the numerical solution can be monitored further by ensuring that Eqs (2.4) hold, since it is clear from physical considerations that they must be satisfied in the improved model also. It only needs to be taken into account that in ρ, φ coordinates the free surface behind the roller corresponds to the segment $\varphi = \pi$. In the given range of parameter values this condition was satisfied to within 10^{-3}.

The results confirm our expectation that the force characteristics of the process do not change significantly when allowance is made for the two-phase nature of the medium. This can be seen from the dashed curves in Fig. 4, which represent the distributions of effective normal stresses and the pressure of the fluid on the contact area obtained by the improved model with $r = 0.3$ and $s = 1$. They are very close to the corresponding solid curves for the classical model. The same was true of the other parameter values. The differences between the integral characteristics of the two models were also correspondingly small. In Fig. 6 the solid curves correspond to the classical model and the dashed curves to the improved model with $r = 0.3$.

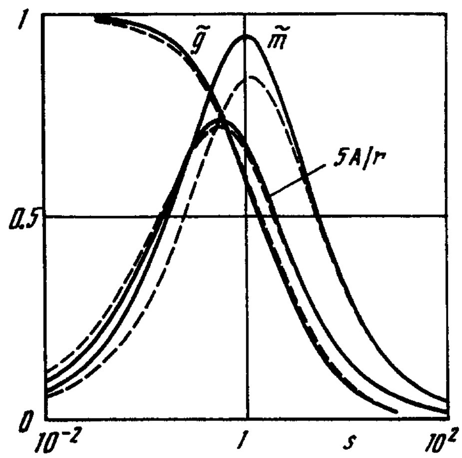

Fig. 6.

Let us examine the position of the two-phase zone in more detail (the dashed curves in Fig. 5). It always lies inside the region of negative pressures of the corresponding classical problem, and also contracts towards the free surface behind the contact area as s increases, degenerating in the limit into the segment $(1, \infty)$ of the real axis. We can establish the position of the boundary of the two-phase zone in the other limiting case, $s \to 0$, as follows.

When $s = 0$, as in the classical formulation, $f_i = -2\rho \sin \varphi$. From the relation between f_i, θ and p and Eqs (7.7) and (7.8), we conclude that the problem for the pressure takes the form

$$-\Delta p - s\frac{\sin 2\varphi}{\rho^2} = 0, \quad p \geqslant 0; \quad \rho = 1: \frac{\partial p}{\partial \rho} = 0; \quad \varphi = 0, \ \pi: p = 0$$

apart from higher-order small terms. The solution p of this problem is independent of ρ

$$p = \frac{s}{4}\begin{cases} \sin 2\varphi - (\varphi / \varphi_*)\sin 2\varphi_*, & 0 < \varphi < \varphi_* \\ 0, & \varphi_* < \varphi < \pi \end{cases} \tag{7.12}$$

Here $\varphi_* = 2.2467$ is found as the root of the equation $2\varphi_* = \operatorname{tg} 2\varphi_*$. Thus, in the w plane the boundary γ is the straight line $\varphi = \varphi_*$, corresponding in the z plane to the hyperbola

$$x^2 \sin^2 \varphi_* - y^2 \cos^2 \varphi_* = \sin^2 \varphi_* \cos^2 \varphi_*$$

In particular, as $s \to 0$ the start of the two-phase zone in the contact area corresponds to the point with coordinate $x_* = -\cos \varphi_* = 0.626$. This is much nearer the outlet than in the classical model, for which the corresponding point $x_* = 0$ was at the middle of the contact area.

It should be noted that this solution is not uniformly valid with respect to z. It is invalid in the neighbourhood of an infinitely distant point, where the boundary layer gives rise to weak convection. We can determine the behaviour of γ at large z and any velocities s by noting that the asymptotic dependence (3.9) still applies in the improved model, so that as $z \to \infty$ the matching conditions on γ take the form

$$x = x_0(y): \ \theta \approx -\frac{qry}{\pi |z|^2} \approx \frac{-qry}{\pi x^2}, \quad \frac{\partial \theta}{\partial y} \approx -\frac{qr(x^2 - y^2)}{\pi |z|^4} \approx -\frac{qr}{\pi x^2} \tag{7.13}$$

allowing for the fact that $x_0 \gg y$. For large $|z|$ the equation for θ in the seepage zone becomes the parabolic boundary-layer equation

$$x < x_0(y): 2s\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2}$$

We can place the beginning of the boundary layer at the point $x = 0$, taking $x_0(0) = 0$, $\theta(0, y) = 0$.

The problem obtained for θ is self-similar. Its solution is sought in the form

$$\theta = \frac{qr}{\pi\sqrt{s}} x^{-3/2}\theta_0(\xi), \quad \xi = \frac{y\sqrt{s}}{\sqrt{x}}$$

Here the boundary γ corresponds to the point $\xi = \xi_0$. To find ξ_0 and $\theta_0(\xi)$ we have

$$\xi > \xi_0: \quad \theta_0'' + \xi\theta_0' + 3\theta_0 = 0 \tag{7.14}$$

$$\xi = \xi_0: \quad \theta_0 = -\xi_0, \quad \theta_0' = -1, \quad \xi \to \infty: \quad \theta_0 = 0 \tag{7.15}$$

We will write the general solution of Eq. (7.14), taking into account the last condition in (7.15) in the form

$$\theta_0 = C(\xi^2 - 1)\exp(-\xi^2/2)$$

The arbitrary constant C and the quantity ξ_0 occurring here are determined by the first two boundary conditions of (7.15). For ξ_0, in particular, we obtain $\xi_0 = \sqrt{(1 + \sqrt{(2)})}$, so that as $z \to \infty$ the boundary γ is the parabola

$$x_0(y) = \frac{s}{1+\sqrt{2}} y^2$$

I wish to thank A. V. Kosterin for initiating the study, for his continued interest and for useful discussions.

This research was supported financially by the Russian Foundation for Basic Research (96-01-00155, 98-01-00200).

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Translated by R.L.